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Higher-order Hamiltonian formalism in field theory

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Abstract. A Hamiltonian formalism is developed from a regular Lagrangian $\mathscr{L}^r$ depending on an arbitrary number of derivatives. The formalism leads to a set of Hamilton equations whose solutions are the same as those of the Euler–Lagrange equations derived from $\mathscr{L}^r$. The Noether currents associated with a symmetry transformation of the Hamiltonian action are also derived.

1. Introduction

It is a well known fact that the generalisation of the ordinary Hamilton variational principle to Lagrangians depending on derivatives of arbitrary order r offers no special difficulty: the Euler–Lagrange equations are obtained in the usual way once the action of the vector fields is defined on the higher-order derivatives which constitute the arguments of the Lagrangian function. When—as we shall do in the present paper—the fibre bundle approach is used, this is accomplished by means of the r-jet prolongation $\bar{X}^r$ of the vector field X, which acts on the coordinates of the bundle $J^r(E)$—the bundle of the r-jets—on which the Lagrangian function is defined (Aldaya and de Azcárraga 1978).

The situation is rather confusing, however, when one tries to obtain a canonical Hamiltonian formalism from a Lagrangian depending on derivatives up to order r, which, through the corresponding modified Hamilton principle, should lead to first-order Hamilton equations equivalent to the Euler–Lagrange ones which in turn would be obtained by applying the ordinary variational principle to that Lagrangian. The initial difficulty is the definition of the Hamiltonian itself. The problem of finding a Hamiltonian formalism for higher-order derivatives has already been considered in analytical mechanics (Borneas 1959, 1969; Koestler and Smith 1965; Krüger and Callebant 1968; Rodrigues and Rodrigues 1970; Mušicki 1978b) and field theory (Borneas 1969; Coelho de Souza and Rodrigues 1969; Rodrigues 1977; these papers contain further references), but not always with the same result. This situation has led us to consider the problem of looking for a *canonical* Hamiltonian formalism for higher-order derivatives. As a result, it will be shown that such a formalism may be constructed on the bundle $J^{1^*}(J^{r-1}(E))$, which will be the space of definition of Hamiltonians. Also, using the fact that $J^r(E)$ may be injected into $J^1(J^{r-1}(E))$, a Hamiltonian formalism will be developed such that the solutions of its corresponding Hamilton equations for suitably defined generalised momenta will coincide with those of the Lagrangian case for higher-order derivatives.

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The paper is organised as follows. In § 2 the problem of defining a higher-order formalism is defined, and its structure analysed. Section 3 is devoted to obtaining such a formalism explicitly. Finally, in § 4 the symmetries of the Hamiltonian action are considered and the associated Noether currents given, showing explicitly the equivalence of the results obtained with those which would have been obtained for the Lagrangian formalism. The paper as a whole will be concerned with field theory, although analytical mechanics may be treated along similar lines.

2. The structure of the Hamiltonian formalism for field theory

A truly Hamiltonian formalism derived from a regular Lagrangian should fulfil two conditions. Firstly, the space of definition of the Hamiltonian density should be parametrised by a system of $m\,.\,n+n+m$ independent coordinates, composed of n field coordinates (coordinates of type q), $m\,.\,n$ momenta (coordinates of type p) and m 'space–time' parameters (coordinates of type t; for Minkowski fields, $m=4$). Secondly, the trajectories of the system described by the Hamiltonian formalism should be obtained from the modified Hamilton variational principle (in which the coordinates of type q and p are varied independently, and which leads to first-order Hamilton equations) and should be in one-to-one correspondence with the critical sections derived from the Lagrangian and the ordinary Hamilton principle.

These two requisites are clearly satisfied in the most simple case—the Hamiltonian formalism associated with a regular Lagrangian depending on first-order derivatives of the fields. In this case, the Lagrangian $\mathscr{L}^1$ is defined as a real function on the bundle of the 1-jets of E, $(J^1(E), M, \pi^1)$, where $E=M\otimes V$, M is the base (usually the Minkowski space) and V is the fibre (which in field theory includes the spin variables of the field). $J^1(E)$ is parametrised by the coordinates $(x^\mu, y^\alpha; g^\alpha_{,\mu})$, and the physical fields are cross sections of E. The *scalar* Hamiltonian density $\mathscr{H}$ is obtained from the regular Lagrangian $\mathscr{L}$ through the Legendre transformation D_{L}, and the space of definition of $\mathscr{H}$ turns out to be (Aldaya and de Azcárraga 1978) $J^{1^*}(E)$—the dual space of the fibre bundle $J^1(E)\to E$—which is parametrised by the coordinates $(x^\mu, y^\alpha; \pi^\mu_\alpha)$. Since $\mathscr{L}$ is regular (i.e. $\det(\partial\mathscr{L}/\partial g^\alpha_{,\mu}\partial g^\beta_{,\nu})\neq 0$), the spaces of critical sections $\mathscr{U}_{\mathscr{L}}$, made up of the solutions of the second-order Euler–Lagrange equations, and $\mathscr{U}'_{\mathscr{H}}$, composed of the solutions of the first-order Hamilton equations

$$\frac{\partial\mathscr{H}}{\partial\pi^\mu_\alpha}=\frac{\partial y^\alpha}{\partial x^\mu},\qquad \frac{\partial\mathscr{H}}{\partial y^\alpha}=-\frac{\partial\pi^\mu_\alpha}{\partial x^\mu}, \tag{2.1}$$

are equivalent.

To generalise the above formalisms to higher order in the derivatives of the fields, higher-order jets are required. We have two different ways of extending the space of definition of first-order Lagrangians $J^1(E)\equiv J^1(J^0(E))$ ($J^0(E)$, the space of the zero-order jets of E, is again E). These correspond to the spaces $J^r(E)$ (of r-jets of E) and $J^1(J^{r-1}(E))$ (of 1-jets of $J^{r-1}(E)$), the first of which may be considered as a subspace of the second (Aldaya and de Azcárraga 1978; Rodrigues 1977). Of these two spaces, only $J^1(J^{r-1}(E))$ admits a dual, $J^{1^*}(J^{r-1}(E))$, suitable for formulating a Hamiltonian theory in accordance with the conditions mentioned above. We shall show that in $J^{1^*}(J^{r-1}(E))$, the momenta may be *canonically* defined, and that there is one momentum corresponding to each derivative of a field variable.

Nevertheless, we wish to construct a higher-order Hamiltonian theory equivalent to the Lagrangian theory, which, when the Lagrangian includes derivatives up to order r, is developed on $J^r(E)$ and not on $J^1(J^{r-1}(E))$. This may be accomplished from the fact that $J^r(E)$ is a subspace of $J^1(J^{r-1}(E))$. In this way, given a Lagrangian $\mathscr{L}^r$ on $J^r(E)$, it is possible to associate with it a Lagrangian $\mathscr{L}^{1,r-1}$ on $J^1(J^{r-1}(E))$ such that the space of critical sections $\mathscr{U}_{\mathscr{L}^{1,r-1}}$ is the same as $\mathscr{U}_{\mathscr{L}^r}$, and then construct on $J^{1^*}(J^{r-1}(E))$ the associated Hamiltonian theory.

We remark that the construction of the formalism on $J^1(J^{r-1}(E))$ may be considered as a generalisation of what is done in analytical mechanics. There (Godbillon 1969), the decomposition of a second-order equation (the Lagrange equation) into two first-order (Hamilton) equations is achieved by imposing on the vector field on $T(M)$ (the phase space with natural coordinates q^i, $\dot{q}^i$), which describes the trajectories, the condition of being simultaneously a cross section of the tangent bundle $\tau(T(M)) = \{T(T(M)), \pi_{T(M)}, T(M)\}$ and of $\tau(\tau(M)) = \{T(T(M)), \pi_M^T, T(M)\}$, the tangent bundle of $\tau(M)$. The projections $\pi_{T(M)}$ and π_M^T which define the two bundles are, in local coordinates,

$$\{q^i, \dot{q}^i, \mathrm{d}q^i, \mathrm{d}\dot{q}^i\} \begin{array}{l} \xrightarrow{\pi_{T(M)}} \{q^i, \dot{q}^i\} \\ \xrightarrow[\pi_M^T]{} \{q^i, \mathrm{d}q^i\}. \end{array}$$

Thus the constraint imposed on the field $X \equiv X^i(\partial/\partial q^i) + Y^i(\partial/\partial \dot{q}^i)$ is given by

$$X^i \equiv \mathrm{d}q^i(X) = \dot{q}^i, \tag{2.2}$$

which, for the corresponding trajectories characterised by the vector fields tangent to them ($X^i = \mathrm{d}q^i/\mathrm{d}t$, $Y^i \equiv \mathrm{d}\dot{q}^i(X) = \mathrm{d}\dot{q}^i/\mathrm{d}t$), gives the condition

$$\dot{q}^i = \mathrm{d}q^i/\mathrm{d}t. \tag{2.3}$$

This shows that a formalism leading to first-order equations (the modified Hamilton variational one on $\tau(T(M))$) may be equivalent to another one leading to second-order equations, provided that a *constraint* is introduced on the trajectories. We shall apply a similar procedure in the field theory case to reduce the formalism on $J^1(J^{r-1}(E))$ to that on $J^r(E)$. Since such a constraint ($\mathrm{d}q^i = \dot{q}^i$ in analytical mechanics) cannot be used directly in the Hamiltonian formalism, this shows that any attempt to obtain a set of first-order Hamilton equations from a formalism on $J^r(E)$ (the space of definition of Lagrangians depending on higher-order derivatives) has to be done through the intermediate steps of the formalisms on $J^1(J^{r-1}(E))$ and $J^{1^*}(J^{r-1}(E))$. This we shall perform explicitly in the next section.

3. Higher-order covariant Hamiltonian formalism

The starting point is a Lagrangian $\mathscr{L}^r$ given as a real function on $J^r(E)$, space which is parametrised by the coordinate system $(x^\mu, y^\alpha, y^\alpha_{\mu_1}, \ldots, y^\alpha_{\mu_1 \ldots \mu_r})$. Thus $\mathscr{L}^r$ is a function of the form

$$\mathscr{L}^r = \mathscr{L}^r(x^\mu, y^\alpha, y^\alpha_{\mu_1}, \ldots, y^\alpha_{\mu_1 \ldots \mu_r}). \tag{3.1}$$

The ordinary Hamilton variational principle leads to the Euler–Lagrange equations

$$\sum_{s=0}^{r} (-)^s \frac{\mathrm{d}^s}{\mathrm{d}x^{\mu_1} \ldots \mathrm{d}x^{\mu_s}} \frac{\partial \mathscr{L}^r}{\partial y^\alpha_{\mu_1 \ldots \mu_s}}\bigg|_{\Psi^r} = 0, \tag{3.2}$$

where the restriction is to cross sections Ψ^r which are r-jet prolongations $\bar{\Psi}^r$, i.e. for which $y^{\alpha}_{\mu_1\ldots\mu_s\nu}=\partial_\nu y^{\alpha}_{\mu_1\ldots\mu_s}$.

A Lagrangian $\mathscr{L}^{1,r-1}$ on $J^1(J^{r-1}(E))$ is a real function (Aldaya and de Azcárraga 1978)

$$\mathscr{L}^{1,r-1}=\mathscr{L}^{1,r-1}(x^\mu, y^\alpha, y^\alpha_{\mu_1},\ldots, y^\alpha_{\mu_1\ldots\mu_{r-1}}; g^\alpha_{,\mu},\ldots, g^\alpha_{\mu_1\ldots\mu_{r-1},\mu}) \tag{3.3}$$

of the coordinates of $J^1(J^{r-1}(E))$. (In the familiar notation of analytical mechanics, the y's correspond to the q's, and the g's to the $\dot{q}$'s.) The ordinary Hamilton principle leads to the equations

$$\frac{\partial\mathscr{L}^{1,r-1}}{\partial y^\alpha_{\mu_1\ldots\mu_s}}-\frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial\mathscr{L}^{1,r-1}}{\partial g^\alpha_{\mu_1\ldots\mu_s,\mu}}\right)=0, \qquad s=0,1,\ldots,(r-1), \tag{3.4}$$

where the 1-jet prolongation condition is fulfilled, i.e. $g^\alpha_{\mu_1\ldots\mu_s,\mu}=\partial_\mu y^\alpha_{\mu_1\ldots\mu_s}$. When the Lagrangians $\mathscr{L}^r[\mathscr{L}^{1,r-1}]$ satisfy the corresponding regularity conditions (Aldaya and de Azcárraga 1978), the modified variational Hamilton principle, where all $y^\alpha_{\mu_1\ldots\mu_s}[y^\alpha_{\mu_1\ldots\mu_s}, g^\alpha_{\mu_1\ldots\mu_s,\mu}]$ vary independently, leads again to the same equations (3.2) [(3.4)].

Let P now be the projection $J^1(J^{r-1}(E))\xrightarrow{P}J^r(E)$ defined by the equations

$$\begin{aligned}
&x^\mu(P(\Psi^{1,r-1}))=x^\mu(\Psi^{1,r-1}),\\
&y^\alpha(P(\Psi^{1,r-1}))=y^\alpha(\Psi^{1,r-1}),\\
&y^\alpha_\mu(P(\Psi^{1,r-1}))=g^\alpha_{,\mu}(\Psi^{1,r-1}),\\
&\quad\vdots \qquad\qquad\qquad \vdots\\
&y^\alpha_{\mu_1\ldots\mu_{r-1}\mu}(P(\Psi^{1,r-1}))=g^\alpha_{\mu_1\ldots\mu_{r-1},\mu}(\Psi^{1,r-1}),
\end{aligned} \tag{3.5}$$

where $\Psi^{1,r-1}$ is a point of $J^1(J^{r-1}(E))$ and $P(\Psi^{1,r-1})$ its projection on $J^r(E)$. The dual application P^* allows us to inject functions on $J^r(E)$ into functions on $J^1(J^{r-1}(E))$, and in this way to define the Lagrangian $\mathscr{L}'^{1,r-1}$ associated with $\mathscr{L}^r$ by

$$\mathscr{L}'^{1,r-1}\equiv P^*\mathscr{L}^r, \tag{3.6}$$

which is thus a function of the form

$$\mathscr{L}'^{1,r-1}=F(x^\mu, y^\alpha, \hat{y}^\alpha_{\mu_1},\ldots,\hat{y}^\alpha_{\mu_1\ldots\mu_{r-1}}; g^\alpha_{,\mu},\ldots, g^\alpha_{\mu_1\ldots\mu_{r-1},\mu}) \tag{3.7}$$

where, as usual, the ^ indicates the missing arguments. We may now apply equations (3.4) to $\mathscr{L}^r$ by using (3.7) instead and incorporating the constraints induced by (3.5). This may be done by means of the Lagrange multipliers

$$\mathscr{L}^{1,r-1}=\mathscr{L}'^{1,r-1}-\sum_{s=0}^{r-1}\lambda^{\mu_1\ldots\mu_s}_\alpha(y^\alpha_{\mu_1\ldots\mu_s}-g^\alpha_{\mu_1\ldots\mu_{s-1},\mu_s}), \tag{3.8}$$

and, as a result, (3.2) is recovered. For instance, for $r=2$

$$\mathscr{L}^{1,r-1}=\mathscr{L}'^{1,r-1}-\lambda^\mu_\alpha(y^\alpha_\mu-g^\alpha_{,\mu}) \tag{3.9}$$

and (3.4) gives on cross sections (after using the constraint, $\partial\mathscr{L}'^{1,r-1}/\partial g^\alpha_{\mu,\nu}=\partial\mathscr{L}^2/\partial y^\alpha_{\mu\nu}$, etc)

$$-\lambda^\mu_\alpha-\frac{\mathrm{d}}{\mathrm{d}x^\nu}\left(\frac{\partial\mathscr{L}^2}{\partial y^\alpha_{\mu\nu}}\right)=0, \qquad \frac{\partial\mathscr{L}^2}{\partial y^\alpha}-\frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial\mathscr{L}^2}{\partial y^\alpha_\mu}+\lambda^\mu_\alpha\right)=0, \tag{3.10}$$

which, after eliminating λ, reproduce (3.2) for $r=2$. In the general case (3.2) is obtained with

$$\lambda_\alpha^{\mu_1\ldots\mu_s\mu}=\sum_{m=1}^{r-s}(-)^m\frac{\mathrm{d}^m}{\mathrm{d}x^{\nu_1}\ldots\mathrm{d}x^{\nu_m}}\frac{\partial\mathscr{L}^r}{\partial y^\alpha_{\mu_1\ldots\mu_s\mu\nu_1\ldots\nu_m}},\qquad s=0,1,\ldots,(r-1),\tag{3.11}$$

which gives $\lambda_\alpha^{\mu_1\ldots\mu_{r-1}\mu}=0$ ($\mathscr{L}^r\neq\mathscr{L}^r(y^\alpha_{\mu_1\ldots\mu_{r-1}\mu\nu_1\ldots\nu_m})$, $m=1,\ldots$); $\lambda_\alpha\equiv-\partial\mathscr{L}/\partial y^\alpha$.

We are now in a position to develop the Hamiltonian formalism. Irrespective of the origin of $\mathscr{L}^{1,r-1}$, it is possible to obtain a Hamiltonian which will lead to the same set of solutions if the regularity condition is fulfilled, since in this case the Legendre transformation D_{L} is a diffeomorphism between $J^1(J^{r-1}(E))$ and $J^{1^*}(J^{r-1}(E))$. A Hamiltonian $\mathscr{H}^{1,r-1}$ on $J^{1^*}(J^{r-1}(E))$ is a real scalar function of the form

$$\mathscr{H}^{1,r-1}=\mathscr{H}^{1,r-1}(x^\mu,y^\alpha,y^\alpha_{\mu_1},\ldots,y^\alpha_{\mu_1\ldots\mu_{r-1}};\pi_\alpha^{,\mu},\ldots,\pi_\alpha^{\mu_1\ldots\mu_{r-1},\mu}).\tag{3.12}$$

In our case $\mathscr{H}^{1,r-1}$ is determined by D_{L} from $\mathscr{L}^{1,r-1}$ in the usual way,

$$\mathscr{H}^{1,r-1}=\pi_\alpha^{,\mu}g^\alpha_{,\mu}+\ldots+\pi_\alpha^{\mu_1\ldots\mu_{r-1},\mu}g^\alpha_{\mu_1\ldots\mu_{r-1},\mu}-\mathscr{L}^{1,r-1},\tag{3.13}$$

since, with $D_{\mathrm{L}}(\Psi^{1,r-1})=\Psi^{1^*,r-1}\in J^{1^*}(J^{r-1}(E))$, we have

$$x^\mu(\Psi^{1^*,r-1})=x^\mu(\Psi^{1,r-1}),\qquad y^\alpha_{\mu_1\ldots\mu_s}(\Psi^{1^*,r-1})=y^\alpha_{\mu_1\ldots\mu_s}(\Psi^{1,r-1}),$$

$$\pi_\alpha^{\mu_1\ldots\mu_s,\mu}(\Psi^{1^*,r-1})=\frac{\partial\mathscr{L}^{1,r-1}}{\partial g^\alpha_{\mu_1\ldots\mu_s,\mu}}(\Psi^{1,r-1}),\qquad s=0,1,\ldots,(r-1).\tag{3.14}$$

This allows us to construct a *canonical* Hamiltonian formalism. The modified Hamilton principle leads, from the variation of the Hamilton functional

$$I'^*(\Psi^{1^*,r-1})=\int_{\Psi^{1^*,r-1}(M)}\Theta^{1^*,r-1},\tag{3.15}$$

where the Poincaré–Cartan form is given by

$$\Theta^{1^*,r-1}=\sum_{s=0}^{r-1}(\pi_\alpha^{\mu_1\ldots\mu_s,\mu}\,\mathrm{d}y^\alpha_{\mu_1\ldots\mu_s})\times\theta_\mu-\mathscr{H}^{1,r-1}\omega\tag{3.16}$$

and $\theta_\mu=\mathrm{i}_{\partial\mu}\omega$ (ω is the volume element on M), to the first-order Hamilton equations

$$\frac{\partial\mathscr{H}^{1,r-1}}{\partial\pi_\alpha^{\mu_1\ldots\mu_s,\mu}}=\frac{\partial y^\alpha_{\mu_1\ldots\mu_s}}{\partial x^\mu},\qquad\frac{\partial\mathscr{H}^{1,r-1}}{\partial y^\alpha_{\mu_1\ldots\mu_s}}=-\frac{\partial\pi^{\mu_1\ldots\mu_s,\mu}}{\partial x^\mu},\qquad s=0,1,\ldots,(r-1)\tag{3.17}$$

in the *canonical* variables $\pi_\alpha^{\mu_1\ldots\mu_s,\mu}$ and $y^\alpha_{\mu_1\ldots\mu_s}$. It is clear that equations (3.17) reproduce (3.4).

We may now turn to our Lagrangian (3.8) in order to obtain the higher-order Hamiltonian formalism. Direct application of the definition (3.13) for the conjugate momenta gives on $\Psi^{1^*,r-1}$

$$\pi_\alpha^{\mu_1\ldots\mu_s,\mu}=(\partial\mathscr{L}'^{1,r-1}/\partial g^\alpha_{\mu_1\ldots\mu_s,\mu})+\lambda_\alpha^{\mu_1\ldots\mu_s\mu}=(\partial\mathscr{L}^r/\partial y^\alpha_{\mu_1\ldots\mu_s\mu})+\lambda_\alpha^{\mu_1\ldots\mu_s\mu}$$
$$=\sum_{m=0}^{r-s}(-)^m\frac{\mathrm{d}^m}{\mathrm{d}x^{\nu_1}\ldots\mathrm{d}x^{\nu_m}}\frac{\partial\mathscr{L}^r}{\partial y^\alpha_{\mu_1\ldots\mu_s\mu\nu_1\ldots\nu_m}},\qquad s=0,1,\ldots,(r-1).\tag{3.18}$$

The $\pi_\alpha^{\mu_1\ldots\mu_s,\mu}$ we have just obtained are De Donder's generalised momenta for the covariant canonical equations (3.17) (De Donder 1935; Mušicki 1978a); here they appear as the momenta *associated with the Lagrangian (3.8).* Substituting (3.12) into

(3.17) we find on cross sections

$$g^{\alpha}_{\mu_1\ldots\mu_s,\mu}=\partial_\mu y^{\alpha}_{\mu_1\ldots\mu_s} \tag{3.19}$$

(i.e. the jet prolongation condition), and

$$-\lambda^{\mu_1\ldots\mu_s}_{\alpha}=\frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial\mathscr{L}^r}{\partial y^{\alpha}_{\mu_1\ldots\mu_s\mu}}+\lambda^{\mu_1\ldots\mu_s\mu}_{\alpha}\right),\qquad s=0,1,\ldots,(r-1), \tag{3.20}$$

which reproduces (3.2) since, in writing (3.20), we have used the definition $\lambda_\alpha\equiv-\partial\mathscr{L}/\partial y^\alpha$.

4. Noether currents

One of the advantages of having obtained the Hamilton equations associated with a Lagrangian $\mathscr{L}^r$ depending on derivatives up to order r through the modified Hamilton principle is that we may now proceed to formulate the Noether theorem in a precise way.

The Poincaré form (3.16) is canonically defined on $J^{1^*}(J^{r-1}(E))$ independent of the fact that the π's are given through their values on the images of the points of $J^1(J^{r-1}(E))$ by D_{L} or directly on $J^{1^*}(J^{r-1}(E))$. Let us now turn to consider the invariance of the Hamiltonian action under a symmetry transformation. A vector field $X^{1^*,r-1}$ on $J^{1^*}(J^{r-1}(E))$ is given by the general expression

$$\begin{aligned}X^{1^*,r-1}=&X^\mu\frac{\partial}{\partial x^\mu}+X^\alpha\frac{\partial}{\partial y^\alpha}+\ldots+X^{\alpha}_{\mu_1\ldots\mu_{r-1}}\frac{\partial}{\partial y^{\alpha}_{\mu_1\ldots\mu_{r-1}}}\\&+X^{,\mu}_{\alpha}\frac{\partial}{\partial\pi^{,\mu}_{\alpha}}+\ldots+X^{\mu_1\ldots\mu_{r-1},\mu}_{\alpha}\frac{\partial}{\partial\pi^{\mu_1\ldots\mu_{r-1},\mu}_{\alpha}}.\end{aligned} \tag{4.1}$$

If $X^{1^*,r-1}$ leaves the Poincaré–Cartan form invariant, i.e.

$$L_{X^{1^*,r-1}}\Theta^{1^*,r-1}|_{\Psi^{1^*,r-1}}=0 \tag{4.2}$$

for any section (whether critical or not) $\Psi^{1^*,r-1}\in(J^{1^*}(J^{r-1}(E)))$, then

$$\mathrm{d}(i_{X^{1^*,r-1}}\Theta^{1^*,r-1})|_{\Psi^{1^*,r-1}}=0 \tag{4.3}$$

on any *critical* section. The formula (4.3) establishes the Noether theorem and leads to the conserved current

$$\begin{aligned}i_{X^{1^*,r-1}}\Theta^{1^*,r-1}=&\sum_{s=0}^{r-1}(\pi^{\mu_1\ldots\mu_s,\mu}_{\alpha}X^{\alpha}_{\mu_1\ldots\mu_s}\theta_\mu-\pi^{\mu_1\ldots\mu_s,\mu}_{\alpha}\,\mathrm{d}y^{\alpha}_{\mu_1\ldots\mu_s}\times i_{X^{1^*,r-1}}\theta_\mu)\\&-\mathscr{H}^{1^*,r-1}X^\mu\theta_\mu,\end{aligned} \tag{4.4}$$

which on cross sections gives

$$\begin{aligned}i_{X^{1^*,r-1}}\Theta^{1^*,r-1}=&\sum_{r=0}^{s-1}[\pi^{\mu_1\ldots\mu_s,\mu}_{\alpha}X^{\alpha}_{\mu_1\ldots\mu_s}\theta_\mu-\pi^{\mu_1\ldots\mu_s,\mu}_{\alpha}\partial_\nu y^{\alpha}_{\mu_1\ldots\mu_s}(X^\nu\theta_\mu-\delta^\nu_\mu X^\sigma\theta_\sigma)]\\&-\mathscr{H}^{1^*,r-1}X^\mu\theta_\mu.\end{aligned} \tag{4.5}$$

Thus the conserved current for the Hamiltonian formalism is given by

$$j^{\mu}=\sum_{s=0}^{r-1}\pi_{\alpha}^{\mu_1\ldots\mu_s,\mu}(X^{\alpha}_{\mu_s\ldots\mu_s}-\partial_\nu y^{\alpha}_{\mu_1\ldots\mu_s}X^{\nu})+\left(\sum_{s=0}^{r-1}\pi_{\alpha}^{\mu_1\ldots\mu_s,\nu}\partial_\nu y^{\alpha}_{\mu_1\ldots\mu_s}-\mathscr{H}^{1^*,r-1}\right)X^{\mu}. \tag{4.6}$$

Using (3.13), (3.14) and (3.18) the current (4.6) may also be written in the form

$$j^{\mu}=\mathscr{L}^{r}X^{\mu}+\sum_{s=0}^{r-1}(X^{\alpha}_{\mu_1\ldots\mu_s}-y^{\alpha}_{\mu_1\ldots\mu_s\nu}X^{\nu})\left(\frac{\partial\mathscr{L}^{r}}{\partial y^{\alpha}_{\mu_1\ldots\mu_s\mu}}+\lambda_{\alpha}^{\mu_1\ldots\mu_s\mu}\right), \tag{4.7}$$

which may be shown to be the current which is obtained from $\mathscr{L}^r$ (or $\mathscr{L}^{1,r-1}$) for the same symmetry operation. This completes the proof of the equivalence of the variational formalisms constructed on $\mathscr{L}^{1,r-1}$ and $\mathscr{H}^{1^*,r-1}$, and, through the connection between $\mathscr{L}^r$ and $\mathscr{L}^{1,r-1}$, exhibits the relation between the Hamiltonian and the Lagrangian generalised formalisms for higher-order derivatives.

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